Math 525

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1 Characteristic Functions

We have talked about moment generating functions:

$$M_X(t) = \mathbb{E}\left(()\, e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x) dx.$$

Sometimes it only converges only when $t = 0$.

<u>Characteristic functions</u>:

$$\phi_X(t) = \mathbb{E}\left(e^{itX}\right) = \int e^{itX} f_X(x) dx.$$

Using Euler's formula

$$e^{itx} = \cos tx + i \sin tx,$$

we notice that $|e^{itx}| = \sqrt{\cos^2 + \sin^2} = 1$.

Hence $\phi_X(t)$ converges in the sense that

$$\begin{aligned}
\phi_X(t) &= \int e^{itx} f(x) dx \\
|\phi_X(t)| &= \left| \int e^{itx} f(x) dx \right| \\
&\leq \int |e^{itx} f(x)| dx \\
&\leq \int |e^{itx}| f(x) dx \\
&\leq \int f(x) dx = 1.
\end{aligned}$$

Example 1.1.

- $X \sim$ Bernoulli$[p]$,

$$\phi(t) = e^{it\cdot 0}q + e^{it}p = q + pe^{it}.$$

- $X \sim$ Unif$[0,1]$,

$$\phi(t) = \int_0^1 e^{itx}dx = \left.\frac{e^{itx}}{it}\right|_{x=0}^{x=1} = \frac{1}{it}(e^{it}-1) = \frac{\cos t - 1 + i\sin t}{it} = -\frac{\sin t + i(\cos t - 1)}{t}.$$

Notice that

$$\phi(0) = \lim_{t\to 0}\frac{1}{it}(e^{it}-1) = \lim_{t\to 0}\frac{ie^{it}}{i} = 1.$$

- $X \sim$ Exp$[\lambda]$,

$$\begin{aligned}\phi(t) &= \int_0^\infty e^{itx}\lambda e^{-\lambda x}dx \\ &= \int_0^\infty \lambda e^{(it-\lambda)x}dx \\ &= \left.\frac{\lambda e^{(it-\lambda)x}}{(it-\lambda)}\right|_{x=0}^{\infty} \\ &= \frac{\lambda}{\lambda - it}\end{aligned}$$

Compare to the generating function

$$M_X(t) = \left.\frac{\lambda e^{(t-\lambda)x}}{(t-\lambda)}\right|_{x=0}^{\infty} = \begin{cases}\frac{\lambda}{\lambda-t} & t < \lambda \\ \text{diverges} & \text{otherwise.}\end{cases}$$

Notice that:

$$\lim_{t\to\pm\infty} = 0.$$

And that is a general fact of characteristic functions.

Brief interlude on integration:

(1) convenience/convergence: unify discrete and continuous RV's

(2) Have to address limits of RV's

  (a) Law of large numbers

  (b) CLT (DeMoire)

Learn measure theory my friend! Check out Math 597.

Example 1.2. Series of functions that converges pointwise but not uniformly:

$$f_n(x) = \begin{cases} 4n^2x & x0 \leq x \leq \frac{1}{2n}, \\ 4n - 4n^2x & \frac{1}{2n} \leq x \leq \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

as in Figure 1.

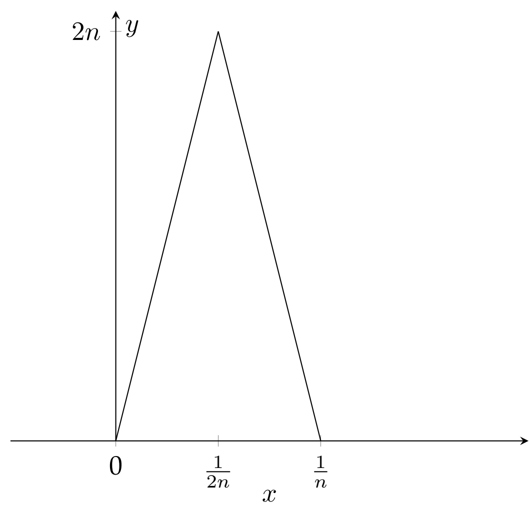

Figure 1: Graph for $f_n(x)$

Let $f(x) = 0$. So pointwise we have

$$\forall x, \delta, \exists N\ s.t.\ \forall n > N, |f_n(x) - f(x)| \leq \delta.$$

However,

$$\lim_{n\to\infty} \int f_n(x)dx = 1 \neq 0 = \int f(x)dx.$$