Math 525

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Nov 3

1 Order Statistics

Suppose

$$X_1, X_2, \ldots, X_n \quad iid's, \sim X.$$

e.g. $X \sim$ uniform$[0, 1]$.

We have a set called "order statistics":

$$\{X_{(1)}(w), X_{(2)}(w), \ldots, X_{(n)}(w)\}$$

where

$$X_{(1)}(w) \leq X_{(2)}(w) \leq \ldots \leq X_{(n)}(w).$$

e.g. $n = 3$, $X_{(2)}(w)$ is actually the median. The range(spread) is $X_{(n)} - X_{(1)}$.

Simplify: since X_i are iid's, we have

$$f_{X_1, X_2, \ldots, X_n}(w_1, w_2, \ldots, w_n) = f_{X_1}(w_1) f_{X_2}(w_2) \ldots f_{X_n}(w_n).$$

Example 1.1. Order statistics when $n = 2$ and $X_i \sim$ uniform: We notice that every point

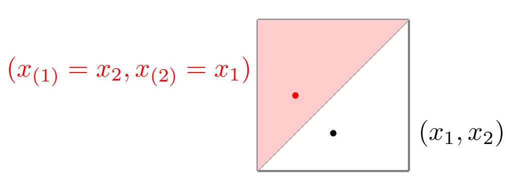

Figure 1: 2-dimensional case

in the lower triangle in Figure 1 will corresponds to a point in the upper triangle in red by symmetry. there are for a certain $f_{X_{(1)},X_{(2)}}(x_{(1)},x_{(2)})$, we have

$$f_{X_{(1)},X_{(2)}}(x_{(1)},x_{(2)}) = \begin{cases} 0, & x_{(1)} > x_{(2)} \\ 2! f_{X_1,X_2}(x_{(1)},x_{(2)}), & x_{(1)} \le x_{(2)}. \end{cases}$$

When $n = 3, 3!$ shuffles as see in Figure 2. There will be six such corners that got transformed into the corner bounded by the red planes.

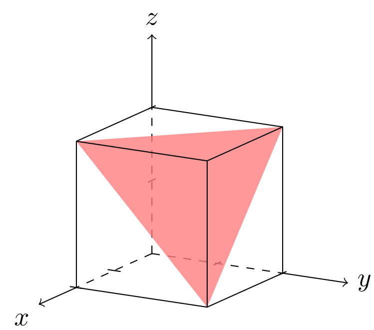

Figure 2: 3-dimensional case

<u>FACT</u>: shuffles preserve volume (in all dimensions.)

For any shuffle, the transformation matrix has the form

$$T = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

which is a square matrix with 1 appearing exactly once on each row and column (0 anywhere else), so $|\det T| = 1$.

<u>BONUS FACT:</u> (Linear algebra) Determinant of a matrix is linear and alternating in columns, i.e. the determinant of A', which is obtained by switching two columns of matrix A, is $-\det A$. So we can switch columns of T to make the 1's all on the diagonal of matrix (which makes an identity matrix.) So $\det T$ is either -1 or 1 depends on whether the number of switching is even or odd.

Proposition 1.1. *Given n iid's $X_1, \ldots, X_n$, the probablity density function of the order statistic is*

$$f_{X_{(1)},X_{(2)},\ldots,X_{(n)}}(x_{(1)},x_{(2)},\ldots,x_{(n)}) = n! \chi_{T_n}(x_{(1)},\ldots,x_{(n)}) \prod_{i=1}^{n} f_{X_i}(x_i)$$

where

$$\chi_{T_n}(x_{(1)}, \ldots, x_{(n)}) = \begin{cases} 1, & x_{(1)} \leq \ldots \leq x_{(n)} \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. See above and homework 8.1 (Grimett 4.14.21). ■

<u>QUESTION</u>: What are the distributions of $X_{(1)}, \ldots, X_{(n)}$ (marginals) ?

<u>ANSWER</u>:

$$f_{X_{(i)}}(x) = \binom{n}{i-1, 1, n-i} F_X(x)^{i-1} f_X(x)(1 - F(x))^{n-i}$$

Proof. As shown in Figure 3, we have $\binom{n}{i-1,1,n-i}$ possible arrangements. So the probabil-

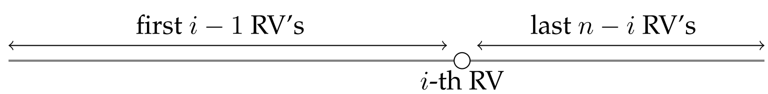

Figure 3: Different arrangements of X_i

ity for the first $i-1$ RV's to be smaller than x is $F(x)^{i-1}$, since all X_i's are independent. Similarly the probability for the last $n-i$ RV's to be larger than x is $(1-F(x))^{n-i}$. ■

Example 1.2. $n = 2, X \sim$ uniform$[0, 1]$. Then we have

$$f_{X_{(1)}}(x) = 2 \cdot x^0 \cdot 1 \cdot (1-x)^1 = 2(1-x),\ f_{X_{(2)}}(x) = 2 \cdot x^1 \cdot 1 \cdot (1-x)^0 = 2x.$$

See Figure 4, we have that

$$\mathbb{E}\left(X_{(1)}\right) = \int_0^1 2x(1-x) = \frac{1}{3},\ \mathbb{E}\left(X_{(2)}\right) = \int_0^1 2x^2 = \frac{2}{3}.$$

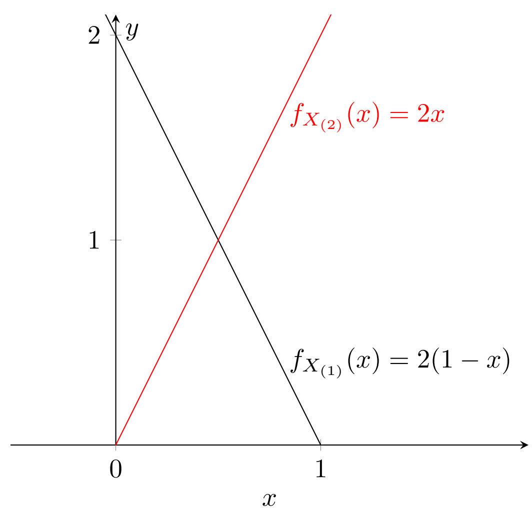

Figure 4: Density functions for $f_{X_{(i)}}$