Math 525

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1 Generating function

Recall two kinds of generating functions:

1. $$G_X(s) = \sum_{n=0}^{\infty} p_X(n) s^n = \mathbb{E}\left(s^X\right)$$

   where X is a "counting" variable (taking values 0, 1, 2, . . .).

2. $$M_X(t) = \mathbb{E}\left(e^{tX}\right).$$

   is a more general form.

Let's compute some examples.

Example 1.1. • $X \sim$ Poisson[λ]:

$$\begin{aligned} G_X(s) &= \sum_{n=0}^{\infty} e^{x} \frac{\lambda^n}{n!} s^n \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda s)^n}{n!} \\ &= e^{-\lambda} e^{\lambda s} \\ &= e^{\lambda(s-1)}. \end{aligned}$$

- $M_X(t)$ for $X \sim \text{Norm}[\mu, \sigma^2]$. Take $\mu = 0, \sigma = 1$.

$$\begin{aligned}
\mathbb{E}\left(e^{tX}\right) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-x^2/2} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2-2xt)/2} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2+t^2/2} dx \\
&= e^{t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2} dx \\
&= e^{t^2/2}.
\end{aligned}$$

Now consider $M_{aX+b}(t)$ since $\sigma Z + \mu \sim$Norm$[\mu, \sigma^2]$.

$$\begin{aligned}
M_{X+b}(t) &= \mathbb{E}\left(e^{tX+b}\right) \\
&= \mathbb{E}\left(e^{tX} \cdot e^{tb}\right) \\
&= e^{tb}\mathbb{E}\left(e^{tX}\right) = e^{tb} M_X(t), \\
M_{aX}(t) &= \mathbb{E}\left(e^{taX}\right) \\
&= \int_{-\infty}^{\infty} e^{tax} f_X(x) dx \\
&= M_X(at).
\end{aligned}$$

Hence

$$M_{\sigma Z+\mu} = M_{\sigma Z}(t) e^{t\mu} = M_Z(\sigma t) e^{t\mu} = e^{t\mu + \frac{\sigma^2 t^2}{2}}.$$

<u>FACTS</u>: G_X, M_X determine the density "uniquely".

<u>CAUTION</u>: "Random sums"

$$S_n = \sum_{i=1}^{n} X_i$$

where X_i are iid's. Then

$$G_{S_m}(S) = (G_{X_1}(s))^n.$$

A random sum is defined as follows:

$$S_n = \sum_{i=1}^{N} X_i, \quad N \text{ is a "counting" random variable.}$$

where X_i are iid's. Do we have

$$G_{S_N}(s) = (G_{X_1}(s))^N?$$

No, there is a variable outside on the RHS. We do have

$$\begin{aligned}
G_{S_N}(s) &= \mathbb{E}\left(s^{S_N}\right) \\
&= \mathbb{E}\left(\mathbb{E}\left(s^{S_N} \mid N\right)\right) \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left(s^{S_N} \mid N = n\right) \mathbb{P}(N = n) \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left(s^{S_n}\right) \mathbb{P}(N = n) \\
&= \sum_{n=0}^{\infty} (G_{X_1}(s))^n \mathbb{P}(N = n) \\
&= G_N(G_{X_1}(s)).
\end{aligned}$$

Example 1.2. Suppose $X_i \sim$ Bernoulli[p] and $N \sim$ Poisson[λ]. What is $\sum_{i=1}^{N} X_i$? Recall:

$$G_{X_1}(s) = q + ps, G_N(s) = e^{\lambda(s-1)}.$$

Hence

$$G_{S_N}(s) = e^{\lambda(q+ps-1)} = e^{\lambda p(s-1)}.$$

So, $\sum_{i=1}^{N} X_i \sim$ Poisson[λp].

2 Applications to random walk

Can be biased: $p \neq \frac{1}{2}$. Is it transient? recurrent?

TRANSIENT: starts at 0, not recurrent.

RECURRENT: starts at 0, $\text{Prob}(\text{return}) = 1$

Theorem 2.1. *A random walk is recurrent if and only if $p = q$.*

Proof. Let

$$p_0(n) = \text{Prob}(S_n = 0), f_0(n) = \text{Prob}(S_n = 0, S_1, \ldots, S_{n-1} \neq 0).$$

and $P_0(s), F_0(S)$ be the generating functions.

Let E = event "$S_n = 0$". $\mathrm{Prob}(E) = p_o(n)$. $F^{(k)}$ = event "k is the first return."

$$\begin{aligned} P(E) &= \sum_{k=1}^{n} \mathbb{P}(E \mid F^{(k)})\mathbb{P}(F^{(k)}) \\ &= \sum p_0(n-k) f_0(k) \\ \xLeftrightarrow{\text{multi by } s^n \text{, add 1}} P_0(s) &= 1 + P_0(s) F_0(s) \end{aligned}$$

So we have

$$P_0(s) = 1 + P_0(s) F_0(s) \tag{1}$$

<u>CLAIM</u>: $P_0(s) = (1 - 4pqs^2)^{-\frac{1}{2}}$.

$$\mathrm{Prob}(S_n = 0) = \begin{cases} \binom{n}{\frac{1}{2}n}(pq)^{n/2} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

$$P_0(s) = \sum_{k=0}^{\infty} \binom{2k}{k} (pq)^k s^{2k}$$

Check: Taylor series of $(1 - 4pqs^2)^{-\frac{1}{2}}$ at $s = 0$. Solve (Equation 1) we have that

$$F_0(s) = [1 - 4pqs^2]^{1/2}.$$

Let $s = 1$, we have

$$F_0(1) = 1 - |p - q|. \qquad \blacksquare$$