Math 525

Yiwei Fu

Oct 27

1 Generating Functions

$$G_X(s) = \sum \mathbb{P}(X = n)s^n = \mathbb{E}\left(s^X\right)$$

If you have to RV's X, Y and $G_X = G_Y$, then $X \sim Y$. They have the same pmf.

Other generating function is moment generating function. Recall the n-th momenet of X is $\mathbb{E}\left(X^n\right)$.

$$\begin{aligned} M_t(X) &= \mathbb{E}\left(e^{tX}\right) \\ &= \mathbb{E}\left(\sum \frac{t^n}{n!} X^n\right) \\ &= \sum \frac{t^n}{n!} \mathbb{E}\left(X^n\right) \end{aligned}$$

when converges.

We have $\frac{de^t}{dt} = e^t$. Hence

$$\frac{d^n}{dt^n} M_X(t)\Big|_{t=0} = \mathbb{E}\left(X^n\right).$$

Example 1.1. Bernoulli$[p] \sim X$.

$$G_X(s) = \mathbb{P}(X = 0)s^0 + \mathbb{P}(X = 1)s^1 = (1 - p) + ps.$$

$$\begin{aligned} M_X(t) = \mathbb{E}\left(e^{tX}\right) &= \sum_x e^{tx}\mathbb{P}(X = x) \\ &= e^{t\cdot 0}(1 - p) + e^{t\cdot 1}p \\ &= (1 - p) + e^t p. \end{aligned}$$

$X \sim B[n, p], X \sim X_1 + \ldots + X_n, X_i$ are iid's and $X_i \sim$ Bern$[p]$.

Theorem 1.2. *Let X, Y be independent RV's in values $\mathbb{Z}_{\geq 0}$ then*

$$G_{X+Y}(x) = G_X(s)G_Y(s).$$

Proof. Just compare coefficients.

$$\mathbb{P}(X+Y=n) = \sum_{i=0}^{n} \mathbb{P}(X=i)\mathbb{P}(Y=n-i)$$

■

Theorem 1.3. *(Abel's Theorem) If $\sum_{n\geq 0} a_n s^n$ with $a_i \geq 0$ and converges for $|s| < 1$ then*

$$\sum a_i = \lim_{s\uparrow 1^-} G_X(s).$$

Now we look at moment generating functions: X, Y are independent RV's, then

$$M_{X+Y}(t) = M_X(t)M_Y(t).$$

Proof.

$$\begin{aligned} M_{X+Y}(t) &= \mathbb{E}\left(e^{t(X+Y)}\right) \\ &= \mathbb{E}\left(e^{tX}e^{tY}\right) \\ &= \mathbb{E}\left(e^{tX}\right)\mathbb{E}\left(e^{tY}\right) \\ &= M_X(t)\cdot M_Y(t) \end{aligned}$$

■

Example 1.4. Binary, $(M_{X_i}(t))^n$.

$G_X(s)$ where $X \sim$ Geom[p]:

$$\begin{aligned} G_X(s) &= \sum_{n=1}^{\infty} s^n \mathbb{P}(X=n) \\ &= \sum_{n=1}^{\infty} s^n p(1-p)^{n-1} \\ &= ps\sum_{n=1}^{\infty} s^{n-1}(1-p)^{n-1} \\ &= ps\frac{1}{1-s(1-p)} \\ &= \frac{ps}{1-sq} \end{aligned}$$

when it converges $|sq| < 1$.

We can look at joint distributions and functions of them, $p_{X,Y}(x, y), G_{X,Y}(s, t) = \sum_{i,j \geq 0} p_{X,Y}(i, j) s^i t^j$.