Math 525

Yiwei Fu

Oct 22

1 Dist from statistics

Suppose we have $X_1, \dots, X_n$ iid's, and $\bar{x} = \frac{1}{n} X_i$ sample mean. $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$ sample var.

Unbiased estimators:

$$\mathbb{E}\left(\bar{X}\right) = \mathbb{E}\left(X_1\right), \mathbb{E}\left(S^2\right) = \operatorname{Var} X_1$$

What kind of RV's are $\bar{X}, S^2$?

Understand when $X_i \sim N(\mu, \sigma^2)$.

If we know

$$\begin{cases} \mu = \mathbb{E}\left(X_i\right) \\ \sigma^2 = \operatorname{Var} X_i \end{cases}$$

then we know X_i's completely.

Theorem 1.1. *Assume X_i's are iid's $X_i \sim N(\mu, \sigma^2)$, then:*

(i) $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

(ii) $(n-1)\frac{S^2}{\sigma^2} \sim \chi^2_{n-1}$

(iii) $\bar{X}, S^2$ *are independent!*

Proof. (i)

$$\bar{X} = \frac{1}{n} \sum X_i$$

and $\sum X_i \sim N(n\mu, n\sigma^2)$. So $\bar{X} \sim N\left(\mu, \frac{n\sigma^2}{n^2}\right) = N\left(\mu, \frac{\sigma^2}{n}\right)$.

(ii) Uses the symmetry of pdf for $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)$.

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}(z_1^2 + \ldots + z_n^2)}$$

Recall the covariance matrix: $Y_1, \ldots, Y_n$ iid's $\sim N(0,1)$.

$$\mathbf{W} = \begin{bmatrix} W_1 \\ \vdots \\ W_n \end{bmatrix} = A \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} = A\mathbf{Y}.$$

Then

$$f_{\mathbf{W}}(w_1, \ldots, w_n) = \frac{1}{(\sqrt{2\pi})^2} e^{-\mathbf{w}^T (AA^T)^{-1} \mathbf{w}/2}.$$

where $V = AA^T =$ covariance matrix. $V_{i,j} = \mathbb{E}\left(W_i W_j\right) = \mathrm{Cov}(W_i, W_j)$.

Special cases when A is orthogonal:

$$A^T A = \mathrm{id}_n \, .$$

It is the same thing

$$\sum_{i=1}^n W_i^2 = \sum_{i=1}^n Y_i^2$$

where $W_i = AY_i$, A is orthogonal.

$$\sum_{i=1}^n W_i^2 = \mathbf{W}^T \mathbf{W} = (\mathbf{Y}^T A^T) A \mathbf{Y} = \mathbf{Y}^T (AA^T)^T \mathbf{Y} = \mathbf{Y}^T \mathbf{Y}.$$

Back to proof. We use unit normals: replace X_i by $Y_i = \dfrac{X_i - \mu}{\sigma}$. So $Y_i \sim N(0,1)$.

$$\bar{Y} = \frac{1}{n} \sum_{Y_i} \sim N\left(0, \frac{1}{n}\right) \iff \sqrt{n}\bar{Y} \sim N\left(0, 1\right).$$

We have $\sqrt{n}\bar{Y} = \sum a_{1j} Y_j$ where $\sum a_{1j}^2 = 1 = \sigma_{Y_i}^2$.

Complete $\sqrt{n}\bar{Y} := W_1$. Using Gram-Schmidt we can fill a matrix with orthonormal basis.

So A is orthogonal and $\mathbf{W} = A\mathbf{Y}$. We have $\sum W_i^2 = \sum Y_i^2$. We have

$$\begin{aligned}
\sum_{i=2}^{n} W_i^2 &= \sum_{i=1}^{n} Y_i^2 - W_1^2 \\
&= \sum_{i=1}^{n} Y_i^2 - \frac{2}{n}\sum_{i,j} Y_i Y_j + \frac{1}{n^2}\sum_{i}\sum_{j=1}^{n}(Y_j)^2 \\
&= \sum (Y_i - \bar{Y})^2 \\
&= \frac{(n-1)S^2}{\sigma^2}
\end{aligned}$$

To sum up, $\sum_{i=1}^{n} W_i^2 = (n-1)\dfrac{S^2}{\sigma^2}$.

FACT: $\chi^2(d) \sim Z_1^2 + \ldots Z_n^2$.

(iii) ($\bar{X}, S^2$ are independent.)

$$\begin{aligned}
\bar{X} &= aW_1 + b \\
S^2 &= a'(W_2^2 + \ldots + W_n^2) + b'
\end{aligned}$$

while W_1 and $W_2, \ldots, W_n$ are independent.

?FACT:

$$\begin{aligned}
\chi^2(d) &= \Gamma\left(\frac{d}{2}, \frac{1}{2}\right) \\
\chi^2(d) &= Z_1^2 + \ldots + Z_n^2
\end{aligned}$$

We showed that $\chi^2(1) = Z_1 = \Gamma\left[\frac{1}{2}, \frac{1}{2}\right]$.

By induction, we want to show $\chi^2(d) \sim \Gamma\left[\dfrac{d}{2}, \dfrac{1}{2}\right]$.

Lemma 1.1.

$$X \sim \Gamma[\alpha, \lambda],\ Y \sim \Gamma[\beta, \lambda]$$

are independent then $X + Y \sim \Gamma[\alpha + \beta, \lambda]$.

It remains to show:

$$\chi_d^2 = Z_1^2 + \ldots + Z_d^2 \sim \Gamma\left[\frac{d}{2}, \frac{1}{2}\right].$$

$$f_{\chi_d^2}(x) = \begin{cases} \frac{(1/2)^{d/2} x^{d/2-1} \theta^{-2/x}}{\Gamma(d/2)} & x \geq 0 \\ 0, x < 0. \end{cases}$$

Know: $\gamma_1^2 = Z_1^2$.

It suffices to show that

$$\Gamma[\alpha, \lambda] + \Gamma[\beta, \lambda] = \Gamma[\alpha + \beta, \lambda] \quad \text{when independent.}$$

$$\mathbb{E}\,(\lambda) + \mathbb{E}\,(\lambda) = \Gamma[2, \lambda].$$

■