Math 525

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1 Still random walk

1.1 Hitting time

Time of first visit to b

$$f_n(b) = \mathbb{P}(S_n = b, S_i \neq b, i = 0, 1, \dots, n-1). \quad (S_n = 0)$$

Theorem 1.1.

$$f_n(b) = \frac{|b|}{n}\mathbb{P}(S_n = b).$$

$$f_n(b) = \mathrm{Prob}(M_{n-1} = b-1, S_{n-1} = b-1, S_b = b)$$

where

$$M_{n-1} = \max\{S_i \mid 0 \leq i \leq n-1\}.$$

Last week we calculated $\mathbb{P}(M_n \geq r)$.

The last flip should be a head, then

$$f_n(b) = p[\mathrm{Prob}\,(M_{n-1} b - 1, S_{n-1} = b-1) - \mathrm{Prob}(M_{n-1} \geq b, S_{n-1} = b-1)].$$

Recall from last time that

$$\mathrm{Prob}(M_n \geq r, S_n = b) = \begin{cases} \mathrm{Prob}(S_n = b) & b \geq r, \\ \left(\frac{q}{p}\right)^{r-b} \mathrm{Prob}(S_n = 2r - b) & b < r. \end{cases}$$

Then

$$
\begin{aligned}
f_n(b) &= p[\text{Prob}\,(M_{n-1}b-1, S_{n-1}=b-1) - \text{Prob}(M_{n-1}\geq b, S_{n-1}=b-1)] \\
&= p\left[\text{Prob}(S_{n-1}=b-1) - \frac{q}{p}\,\text{Prob}(S_{n-1}=2b-(b-1)))\right] \\
&= \frac{b}{n}\,\text{Prob}\,(S_n=b)
\end{aligned}
$$

Same argument for $b<0 \implies f_n(b) = \frac{|b|}{n}\,\text{Prob}(S_n=b)$.

2 Continuous distributions

X is a random valued RV. Instead of p.m.f $p_X(n)$ we have p.d(ensity).f $f_X(x)$.

Events: e.g. $\{a\leq X\leq b\}; \text{Prob}(a\leq X\leq b) = \int_a^b f_X(x)dx$.

Example 2.1. uniform distribution on $[0,1]$:

$$f_X = \chi_{[0,1]} : \begin{cases} \chi_{[0,1]}(x) = 1 & 0\leq x\leq 1, \\ 0 & \text{else.} \end{cases}$$

Need:

1. $f_X(x)\geq 0$.
2. $\int_{-\infty}^{\infty} f_X(x)dx = 1$.
3. check

If $X\sim$ uniform$[a,b]$, then

$$f_X(x) = \frac{1}{b-a}\chi_{[a,b]}(x).$$

Suppose $X\sim$ uniform$[0,1]$. Distribution function

$$F_X(x) = \text{Prob}(X\leq x), 0\leq F(x)\leq 1, \lim_{x\to-\infty} = 0, \lim_{x\to\infty} = 1.$$

What is the relation of F_x, f_x?

$$F_X(x) = \int_{-\infty}^{x} f_Y(y)dy.$$

By FTC we have

$$\frac{dF_X(x)}{dx} = \frac{d}{dx}\int_{-\infty}^{x} f_X(y)dy = f_X(x)$$

Example 2.2. (Exponential distribution)

$$f_X(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & x \geq 0. \end{cases}.$$