Math 525

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Paths avoiding zero

At end last time:

$N_n^d(a,b) = \#$ of sample paths from 0 to b, length n $s.t.$ NOT revisiting 0 in between $= \frac{b}{n}N(0,b)$.

Then we have

$$\text{Prob}\,(S_1 \dots S_n \neq 0, S_n = b) = \frac{b}{n}\,\text{Prob}\,(S_n = b),$$

which follows from the counting.

If we sum over b's, then we get

$$\mathbb{E}(\text{path that doesn't return to zero in time n}) = \frac{1}{n}\mathbb{E}(|S_n|).$$

Lemma 0.1.

$$N_n^d(a,b) = \frac{|b|}{n}N(0,b)$$

Sublemma: "Reflection principle" Number of paths $N_n^0(a,b) = N_n^0(-a,b), b > a > 0$.

Proof. By picture. ■

Proof. (of lemma)

$$\begin{aligned} b > 0, N_n^d(0,b) &= N_{n-1}(1,b) - N_{n-1}^0(1,b) \\ &= N_{n-1}(1,b) - N_{n-1}(-1,b) \\ (*) &= \frac{b}{n}N_n(0,b). \end{aligned}$$

(*) is an easy combinatorial calculation. ■

Maximum wander?

$$M_n = \max_{0\leq m\leq n} S_n, \quad S_0 = 0.$$

Theorem 0.1.

$$\text{Prob}\,(M_n \geq r, S_n = b) = \begin{cases} \text{Prob}\,(S_n \leq b) & b \geq r, \\ \left(\dfrac{q}{p}\right)^{r-b} \text{Prob}\,(S_n = 2r - b) & r > b. \end{cases}$$

Proof. When $b \geq r$ is OK. When $r > b$, we use reflection again.

$$N_n^r(0,b) = \text{\# of paths from 0 to } b \text{ which visit } r.$$

$$\begin{aligned} \text{Prob}\,(M_n \geq r, S_n = b) &= N_n^r(0,b) p^{1/2(n+b)} q^{1/2(n-b)} \\ &= \left(\frac{q}{p}\right)^{r-b} N_n^r(0,b) p^{1/2(n+2r-b)} q^{1/2(n-2r+b)} \end{aligned}$$

Claim: $N_n^r(0,b) = N_n(0, 2r-b)$.

We use reflection picture to prove this claim.

$$N_n^r(0,b) = N_n(2r-b, b).$$

■

For $p = q = \frac{1}{2}$,

$$\text{Prob}\,(M_n \geq r) = 2\,\text{Prob}\,(S_n \geq r+1) + \text{Prob}\,(S_n = r)$$

starting from $S_0 = 0$.

Next time: hitting time?

(First visit to b)

$$f_n(b) = \text{Prob}\,(S_n = b, S_m \neq b, 0 \leq m \leq n-1) = \frac{|b|}{n} \text{Prob}\,(S_n = b).$$