Math 525

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Spet 17

Expectation

$$\mathbb{E}(x) = \sum_{k=1}^{\infty} kpq^{k-1} = p \sum kq^{k-1}$$

Note that

$$\frac{d}{dx} \sum_{k=1}^{\infty} x^k = \sum_k x^{k-1}$$

i.e.

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$$

Then we get

$$\curvearrowleft = p \frac{1}{(1-q)^2} = \frac{1}{p}$$

Poisson RV: p.d.f

$$p_x(k) = e^{-\lambda} \frac{\lambda^k}{k!}, k \in \mathbb{Z}_+.$$

$$\sum e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$$

Parameter $\lambda > 0$.

Claim: $\mathbb{E}(x) = \lambda$.

Where does this come from?

1. Approximation to the binomial (small k and $pn = \lambda$)
2. occurrences of rare events

Calculate: $X_{n,p} = Bin[n, p]$, expectation = $np = \lambda$

$$
\begin{aligned}
\mathbb{P}(X_{n,p}) &= \binom{n}{k} p^k q^{n-k} \\
&= \frac{n(n-1)\dots(n-k+1)}{k!} p^k (1-p)^{n-k} \\
&= (np)^k \frac{1}{k!} (1) \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{k-1}{n}\right) (1-p)^{n-k} \\
&= \frac{\lambda^k}{k!} [\to 1][?]
\end{aligned}
$$

$$
\begin{aligned}
(1-p)^{n-k} &= \frac{(1-p)^n}{(1-p)^k} \\
&= [(1-p)^{1/p}]^{\lambda} \\
&= (e^{-1/p})^k = e^{-\lambda}
\end{aligned}
$$

First: Jointly distributed RV's. X, Y are RV's of same Ω. $X = $ 1st toss, $Y = $ 2nd toss. X, Y are the "marginals."

p.m.f

$$p_X(i) = \sum_j p_{X,Y}(i,j).$$

$$p_Y(j) = \sum_i p_{X,Y}(i,j).$$

When are X, Y independent?

$$p_{X,Y}(i,j) = p_X(i) p_Y(j).$$

There is a notion related to expectation which is weaker than independence correlation: if

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$$

we say X, Y are uncorrelated.

If X, Y are independent, with $\mathbb{E}(X) = \mathbb{E}(Y) = 0$ then they are uncorrelated.

Basic Fact:

$$\mathbb{E}(X_1 + \dots + X_n) = \mathbb{E}(X_1) + \dots + \mathbb{E}(X_n).$$