Math 525

Yiwei Fu

Spet 13

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G & S, 3rd ed, 1.8.28: if $\mathbb{P}(E) > 0$ then $E \neq \emptyset$.

Agenda: "law of averages"; basics of prob. mass function/prob. density functions, cumulative distribution functions (cdf's).

Idea: compare statistical and model ideas of probability,

Suppose $E \subset \Omega$. Repeat, independently, the experiment n time.

$$n(E) = \# \text{ of these trials such that outcome} \in E.$$

$$\lim_{n\to\infty} \frac{n(E)}{n} = \mathbb{P}(E)\ ?$$

Bernoulli considers:

$$\frac{1}{n} S_n = \frac{1}{n} \sum_{i=1}^{n} I_{E_i}$$

where I_{E_i} is indicator random variable for E in the ith trial.

$$I_{E_i} = \begin{cases} 1 & i\text{-th result} \in E \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbb{P}(E) = \mathbb{P}(E_i) = p.$$

Assume (WLOG) experiment is a biased coin with $\mathbb{P}(H) = p$.

Theorem 1.1.

$$\textit{As } n \to \infty, \mathbb{P}\left(p + \varepsilon > \frac{1}{n} S_n > p - \varepsilon\right) \longrightarrow 1$$

“Large deviations go to zero."

Proof.

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\begin{aligned}
\mathbb{P}\left(\frac{1}{n}S_n > p - \varepsilon\right) &= \mathbb{P}\left(S_n > np - n\varepsilon\right) \\
&= \sum_{k > np + n\varepsilon} \binom{n}{k} p^k q^{n-k} \\
&\leq \sum_{k} e^{\lambda[k - n(p+\varepsilon)]} \binom{n}{k} p^k q^{n-k} \\
&\text{for some } \lambda > 0 \\
&\leq e^{-\lambda n\varepsilon} \sum \binom{u}{k} (pe^{\lambda q})^k (qe^{-\lambda k})^{u-k} \\
&\leq e^{-\lambda n\varepsilon} (pe^{\lambda q} + qe^{-\lambda p})^n
\end{aligned}
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