Math 525

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Two basic lemmas:

1. by cases (partitioning)
2. conditioning (telescoping)

Try something:

1. Partitioning: Say $E_1, \dots, E_n, \dots$ (a countable collection of events) are a pairwise disjoint, exhaustion of Ω $i.e.$ $\forall i \neq j, E_i \cap E_j = \emptyset$ and $\bigcup_i E_i = \Omega$.
   Then for any event A,
   $$\mathbb{P}(A) = \sum_i \mathbb{P}(A \mid E_i)\mathbb{P}(E_i).$$
   The proof is basic from the distributive law from set theory $i.e.$
   $$A = \bigcup_i (A \cap E_i).$$
   The next statement is not true when it is not pairwise disjoint:
   $$\mathbb{P}(A) = \sum_i \mathbb{P}(A \cap E_i) = \sum_i \frac{\mathbb{P}(A \cap E_i)}{\mathbb{P}(E_i)}\mathbb{P}(E_i) = \sum_i \mathbb{P}(A \mid E_i)\mathbb{P}(E_i).$$
   Set-theoretic proof:
   $$\begin{aligned} A &= A \cap \Omega \\ &= A \cap \left(\bigcup_i E_i\right) \\ &= \bigcup_i A \cap E_i \text{ “distributive law."} \end{aligned}$$

2. Telescoping: Suppose a finite number of E_i

$$\begin{aligned}\mathbb{P}(E_1 \dots E_n) &= \mathbb{P}(E_n|E_1 \dots E_{n-1}) \dots \mathbb{P}(E_3|E_2E_1)\mathbb{P}(E_2|E_1)\mathbb{P}(E_1) \\ &= \frac{\mathbb{P}(E_1 \dots E_n)}{\mathbb{P}(E_1 \dots E_{n-1})} \dots \frac{\mathbb{P}(E_1E_2)}{\mathbb{P}(E_1)}\mathbb{P}(E_1).\end{aligned}$$

Example: Suppose $\Omega = u$ coin flips (fair). $E_i = u$ flip are H on i-th flip.

Problems: "2 children":

(a) New neighbors move in next door and they have 2 children. You are told they have at least one daughter. What is the probability they have 2 daughters?

(b) New neighbors move in next door and have 2 children. I meet the father walking with one of the children, who is a daughter. What is the probability they have 2 daughters?

Model:
1 birth: $\mathbb{P}(S) = 0.5, \mathbb{P}(D) = 0.5$. 2 births: assuming independence, 2 single births. $\mathbb{P}(SS) = 0.25, \mathbb{P}(SD) = 0.25, \mathbb{P}(DS) = 0.25, \mathbb{P}(DD) = 0.25$.

(a)

$$\mathbb{P}(2D) = \mathbb{P}(DD|DD \cup DS \cup SD) = \frac{1}{3}.$$

(b)

$$\mathbb{P}(2D) = \frac{1}{2}$$

Problem: Deck of cards, shuffled. Flip 1 by 1, get a first ace. Then, is the 2 of clubs more likely than ace of spades as the next card?