Math 525

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1 Simplest models

Suppose we have a finite $\Omega : |\Omega| < \infty$, all are equally likely.

$$\mathbb{P}(i) = \frac{1}{|\Omega|}.$$

$$E \in \Omega,\ \mathbb{P}(E) = \frac{|E|}{|\Omega|}.$$

Axioms: Easy! (just count the numbers.)

Here are some special cases:
Flip 2 coins, we have $2^2 = 4$ outcomes, we can record the result as a vector $(H/T, H/T)$.
Here we can think of the sample space Ω as a 2-dimensional table:

Notice that the top left and bottom right outcomes are different, and all four outcomes are equally likely.

Suppose $E =$"have at least one H". We have

$$P(E) = \frac{3}{4}.$$

2 Sampling: Balls in urns

An urn with a balls red, b balls white.
Now we have to decide an experimental procedure:

(i) with replacement

(ii) without replacement.

Real life counterparts: sampling population of Michigan when you have a phone book.
Suppose $A =$"draw red", $B =$"draw white".
1st draw:

$$\mathbb{P}(A) = \frac{a}{a+b}.$$

2nd draw: $\mathbb{P}(A)$ would be the same. Thus we have

$$\mathbb{P}(A_1 \cap A_2) = \left(\frac{a}{a+b}\right)^2.$$

This illustrate the idea of independence (formalized later.)

Now say we draw an "A" first (without replacement), then we have

$$\mathbb{P}(A_2 \mid A_1) = \frac{a-1}{a+b-1}$$

where $\mathbb{P}(A_2 \mid A_1)$ is the conditional probability (the probability of A_2 given that A_1 happened.)

Remark. Finite probabilistic problems depend on counting and correctly interpreting the experimental protocol.

Recall the axioms:
$E \subset \Omega$, then

1. $\mathbb{P}(E) \in [0, 1]$
2. $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\Omega)$
3. $\mathbb{P}(\bigcap_i A_i) = \sum_i \mathbb{P}(A_i)$, where i is countable $A_i \cap A_j = \emptyset$ for all i, j.

What about overlap?

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

Refer to Venn diagram.

Formula:

$$\mathbb{P}(\bigcup_i E_i) = \sum_i \mathbb{P}(E_i) - \sum_{i \neq j} \mathbb{P}(E_i \cap E_j) + \sum \mathbb{P}(E_i \cap E_j \cap E_k) - \dots$$

Proof. By induction. **Base case** $n = 2$.

$$A \cup B = (A \setminus (A \cap B)) \cup (B \setminus (A \cap B)) \cup (A \cap B)$$

Where all three sets are disjoint.

$$\begin{aligned}\mathbb{P}(A \cup B) &= \mathbb{P}(A \setminus (A \cap B)) + \mathbb{P}(B \setminus (A \cap B)) + \mathbb{P}(A \cap B)\\ &= \mathbb{P}(A) - \mathbb{P}(A \cap B) + \mathbb{P}(B) - \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B)\\ &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).\end{aligned}$$

The rest follows by induct on n. (refer to textbook) ■

3 Independence & Conditional Probabilities

Consider 2 coin flips, all outcomes equal. Suppose Ω is sample space, $E =$"at least one head".

$$\mathbb{P}(\text{2 Heads} \mid E) = \frac{1}{3}.$$

We can count this, but what is the formula?

$$\mathbb{P}(\text{2 Heads} \mid E) = \frac{\mathbb{P}(\text{2 Heads} \cap E)}{\mathbb{P}(E)} = \frac{\mathbb{P}(\text{2 Heads})}{\mathbb{P}(E)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

Clearly we have $\mathbb{P}(E \mid E) = 1$.

Definition 3.1. Conditional Probability.

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}, (\mathbb{P}(B) \neq 0).$$

Definition 3.2. Independence of event A and B.

$$\mathbb{P}(A \mid B) = \mathbb{P}(A)\ i.e.\ \mathbb{P}(A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \Leftrightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Remark. Bayes Theorem.

$$\begin{aligned}\mathbb{P}(A \mid B) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}\\ &= \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(B)} \frac{\mathbb{P}(A)}{\mathbb{P}(A)}\\ &= \frac{\mathbb{P}(A)}{\mathbb{P}(B)} \mathbb{P}(B \mid A).\end{aligned}$$

Example 3.1. "Compounds experiment" 3 coins, $F : p = \frac{1}{2}, C : p = \frac{2}{3}, B : p = \frac{1}{3}$. If $F = H$ show C, otherwise show B.

Questions: what is the prob of flipping and seeing an H ($= H_1$)?

$$\begin{aligned}\mathbb{P}(H_1) &= \mathbb{P}(H_1 \mid F = H)\mathbb{P}(F = H) + \mathbb{P}(H_1 \mid F = T)\mathbb{P}(F + T) \quad \text{(partitioning into cases or conditioning.)}\\ &= \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2}\\ &= \frac{1}{2}.\end{aligned}$$

What about $H_1 H_2$?