Notes for Math 668 – Combinatorics of $\mathrm{GL}_n \mathbb{C}$ Representation Theory

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FA 2022

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Office hours:

Chapter 1

We'll be studying finite dimensional representation of $\mathrm{GL}_n \mathbb{C}$ and its connections to combinatorics, including symmetric polynomials, Young Tableaux, crystals, JDQ and RSK, webs, standard theory.

not doing, but good topics:

- ∞-dimensional representations
- Cluster algebras
- total positivity
- other Lie groups
- S_n
- $\mathrm{GL}_n \mathbb{F}_p$ representation theory.

All of these use the basic $\mathrm{GL}_n \mathbb{C}$ theory. In addition, $\mathrm{GL}_n \mathbb{C}$ representation theory is very close to $U(n)$-represention theory.

1.1 Introduction

<u>Reminder Time</u> G a group, K a field, V a k-vector space. Then a representation of G on V is an action of G on V by K-linear maps $G \times V \to V, (gh)(v) = g(h(v)), \mathrm{id} \cdot v = v, g(u+v) = g(u) + g(v), g(cv) = cg(v)$.

In other words, a homomorphism $\rho : G \to \mathrm{GL}(V)$. We are looking at $\rho : \mathrm{GL}_n \mathbb{C} \to \mathrm{GL}_N(\mathbb{C})$.

Let $W = \mathbb{C}^n$ with the standard $\mathrm{GL}_n \mathbb{C}$ action. We like:

- $W, W \oplus W$.

- $W^{\otimes k}$, $\bigwedge^k W$ and $\operatorname{Sym}^k W$,
- $W^V = \operatorname{Hom}(W, \mathbb{C})$.
- $\mathbb{C}$, $g \mapsto (\det g)^k$, $k \in \mathbb{Z}$.

Some representations we don't want to study.

- $|\det(g)|^\alpha$, $\alpha \in \mathbb{R}$, not even when $\alpha = 1$.
- $g \mapsto \overline{g}$
- $g \mapsto \sigma(g)$, $\sigma \in \operatorname{Gal}(\mathbb{C}/\mathbb{Q})$.

(They are not algebraic.)

Definition 1.1.1. We say that $\rho : \mathrm{GL}_n \mathbb{C} \mapsto \mathrm{GL}_N(\mathbb{C})$ is *polynomial* if the N^2 matrix entries $\rho(g)_{ij}$ are polynomials in the entries of g, i.e. $\mathbb{C}[g_{pq}]$. We'll say ρ is *algebraic* if the $\rho(g)_{ij} \in \mathbb{C}[g_{pq}, (\det g)^{-1}]$.

Definition 1.1.2. For any group G, and representation V over a field K, we define the character χ_V of V to be the function $G \to K$ given by $\chi_V(g) = \operatorname{tr}(\rho_V(g))$

Notice that

$$\chi_V(hgh^{-1}) = \operatorname{tr}(\rho_V(hgh^{-1})) = \operatorname{tr}(\rho_V(h)\rho_V(g)\rho_V(h)^{-1}) = \operatorname{tr}(\rho_V(g)) = \chi_V(g).$$

The diagonalizable matrices are dense in $\mathrm{GL}_n \mathbb{C}$, so any continuous function is determined by its values on diagonalizable matrices. So a continuous conjugacy invariant function is determined by its values on diagonal matrices.

If V is a polynomial representation, then

$$\chi_V\left(\begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}\right) \in \mathbb{C}[z_1, \dots, z_n].$$

($\mathbb{Z}[z_1, \dots, z_n]$, in fact.)

If V is algebraic,

$$\chi_V\left(\begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}\right) \in \mathbb{C}[z_1^\pm, \dots, z_n^\pm].$$

($\mathbb{Z}_{\geq 0}[z_1^\pm, \dots, z_n^\pm]$, in fact.)

In general,

$$\sigma \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix} \sigma^{-1} = \begin{bmatrix} z_{\sigma(1)} & & \\ & \ddots & \\ & & z_{\sigma(n)} \end{bmatrix}$$

So characters of polynomial/algebraic representations are symmetric polynomials/Laurent polynomials.

Denote $\Lambda_n = \mathbb{Z}[z_1, \ldots, z_n]^{S_n}$, and $\Lambda_n^{\pm} = \mathbb{Z}[z_1^{\pm}, \ldots, z_n^{\pm}]^{S_n}$. Λ is symmetric polynomials in ∞-ly many vars.

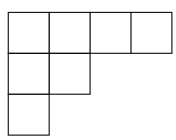
1.2 Symmetric Polynomials

$$\Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}, \Lambda_n^{\pm} = \mathbb{Z}[x_1^{\pm}, \ldots, x_n^{\pm}]^{S_n}$$

Definition 1.2.1. A *partition* is a weakly decreasing sequence of positive integers

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$$

We draw them as configuration of boxes called *Young Diagrams*.



$\leftrightarrow (4, 2, 1).$

French convention is increasing diagrams; Russian convention is angled.

We often pad our partitions with 0's: $(4, 2, 1), (4, 2, 1, 0), \ldots$ The size of a partition $|\lambda| = \sum_j \lambda_j$, which is the number of boxes. The length of a partition $\ell(\lambda) = \#j, \lambda_j > 0$. The transpose of $(4, 2, 1)^T = (3, 2, 1, 1)$. $(\lambda^T)_j = \#i, \lambda_i \geq j$.

Definition 1.2.2. The *dominance order* is the partial order defined by

$$\mu \preceq \lambda \iff \forall i, \mu_1 + \ldots + \mu_i \leq \lambda_i + \ldots + \lambda_i.$$

Definition 1.2.3. Suppose λ a partition, then

$$\Lambda_n \ni m_\lambda(x_1, x_2, \ldots, x_\lambda) = \sum_{(c_1, \ldots, c_n) \in S_n(\lambda_1, \ldots, \lambda_n)} x_1^{c_1} x_2^{c_2} \ldots x_n^{c_n}$$

<u>NOTE</u> We are not counting with multiplicity. All coefficients of m_λ are 0 or 1:

$$m_{1,1}(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3, \neq 2(x_1x_2 + x_1x_3 + x_2x_3).$$

So we have the monomial symmetric functions

$$\Lambda_n = \bigoplus_{\ell(\lambda) \leq n} \mathbb{Z} \cdot m_\lambda.$$

Definition 1.2.4.

$$e_k(x_1, \ldots, x_n) = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1} x_{i_2} \ldots x_{i_k} = m_{1\ldots1}$$

We have

$$e_{\lambda_1 \lambda_2 \ldots \lambda_\ell} = e_{\lambda_1} e_{\lambda_2} \ldots e_{\lambda_\ell}$$

We call e elementary symmetric functions.

Example 1.2.1.

$$\begin{aligned} e_{3,1}(w, x, y, z) &= e_3 e_1 \\ &= (wxy + wxz + wyz + xyz) \cdot (w + x + y + z) \\ &= (w^2xy + \ldots) + 4wxyz \\ &= m_{2,1,1} + 4m_{1,1,1,1}. \end{aligned}$$

Notice $(2, 1, 1)$ is the transpose of $(3, 1)$.

Lemma 1.2.1.

$$e_\lambda = m_{\lambda^T} + \textit{linear combination of } m_\mu \textit{ with } m_\mu \prec \lambda^T$$

Proof. Suppose m_μ occurs with positive coefficients in $e_\lambda = e_{\lambda_1} \ldots e_{\lambda_\ell}$. Then $x_1^{\mu_1} \ldots x_n^{\mu_n}$ occurs in the expansion.

$$\begin{aligned} \mu_1 &\leq \#\{i : \lambda_i \geq 1\} = \ell(\lambda) = (\lambda^T)_1 \\ \mu_1 + \mu_2 &\leq 2\#\{i : \lambda_i \geq 2\} + \#\{i : \lambda_i = 1\} \\ &\leq \#\{i : \lambda_i \geq 2\} + \#\{i : \lambda_i \geq 1\} \\ &\leq (\lambda^T)_2 + (\lambda^T)_1 \\ &\vdots \end{aligned}$$

For any j,

$$\begin{aligned}\mu_1+\mu_2+\ldots+\mu_j &\le j\cdot\#\{i:\lambda_i\ge j\}+\sum_{k=1}^{j-1}k\#\{i:\lambda_i=k\}\\ &\le\sum_{k=1}^{j}\#\{i,\lambda_i\ge k\}=\sum_{k=1}^{j}(\lambda^T)_k\end{aligned}$$

So we have $\mu\preceq\lambda^T$. To get equality, must take $x_1x_2\ldots x_{\lambda_i}$ as the conjugation from e_λ, then we have the term m_{λ^T} with coefficient 1. ■

Corollary 1.2.1.

$$\Lambda_n=\bigoplus_{\ell(\lambda^T)\le n}\mathbb{Z}e_\lambda=\bigoplus_{\ell(\lambda)\le n}\mathbb{Z}e_{\lambda^T}.$$

Proof. Fix a degree d. The e_{λ^T} with $|\lambda|=d$ are related to m_λ with $|\lambda|=d$ by an upper triangular matrix. ■

$$\Lambda_n=\mathbb{Z}[e_1,e_2,\ldots,e_n]$$

$\Lambda_n\to\Lambda_{n-1}$ by setting $x_n\mapsto 0$. Equivalently

$$m_\lambda\mapsto\begin{cases}m_\lambda & \ell(\lambda)\le n-1\\ 0 & \ell(\lambda)\ge n\end{cases},e_\lambda\mapsto\begin{cases}e_\lambda & \ell(\lambda^T)\le n-1\\ 0 & \ell(\lambda^T)\ge n\end{cases}$$

$\Lambda=\lim_{J\in n}\Lambda_n$ graded inverse limit.

In any fixed degree, this diagram stabilizes, with

$$\Lambda=\mathbb{Z}[e_1,e_2,\ldots]=\bigoplus_\lambda\mathbb{Z}\cdot e_\lambda=\bigoplus\mathbb{Z}\cdot m_\lambda$$

We can obtain a lot of equations that does not consider how many variables we use, like $m_1^2=m_2+2m_{11}$, $m_1^3=m_3+3m_{21}+6m_{111}$.

Λ is a graded ring with ring maps to every Λ_n. We would have diagrams like

$$\cdots\to\Lambda_3\xrightarrow{x_3\mapsto 0}\Lambda_2\xrightarrow{x_2\mapsto 1}\Lambda_1$$

Concretely, $\Lambda=\mathbb{Z}[e_1,e_2,e_3,\ldots]$, and the maps $\Lambda\to\Lambda_n$ sends $e_i\mapsto e_i$ for $i\le n$ and e_jto0 for $j>n$.

$\Lambda = \bigoplus_\lambda \mathbb{Z} \cdot e_\lambda = \bigoplus_\lambda \mathbb{Z} \cdot m_\lambda$. For $e_\Lambda \cdot e_\mu$, do the computation in a large enough Λ_n.

Given $\to \cdots \to x_3 \to x_2 \to x_1$. We say that Y is $\lim_{\infty \leftarrow n} X_n$ if there are maps $\pi_1 : Y \to x_1, \pi_2 : Y \to X_2, \dots$ and for any Z we such compatible maps $(\alpha_1, \alpha_2, \dots)$, there exists unique $f : Z \to Y$ such that $\alpha_i = \pi_i \circ f$.

e_k is the character of $\bigwedge^k \mathbb{C}^n$.

$h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots = \sum_{i_1 \le i_2 \le \dots \le i_k} x_{i_1} x_{i_2} \dots x_{i_k}$, which is the character of $\mathrm{Sym}^k \mathbb{C}^n$.

$h_2 = \sum_{i \le j} x_i x_j = \sum x_i^2 + \sum_{i<j} x_i x_j = m_2 + m_{11}$

$h_{11} = h_1^2 = (\sum x_i)^2 = m_2 + 2m_{11}$.

We already know that $\Lambda = \mathbb{Z}[e_1, e_2, \dots] = \bigoplus_\lambda \mathbb{Z} \cdot e_\lambda$. We want to know that if $\Lambda = \mathbb{Z}[h_1, h_2, \dots] = \bigoplus_\lambda \mathbb{Z} \cdot h_\lambda$.

The key thing is to check that e_k's are polynomials of h_k's and vice versa. Once we do that, we know the e's and the h's generate the subring of Λ.

Lemma 1.2.2. *For any k, $\sum_{j=0}^k (-1)^j e_j h_{k-j} = 0$.*

Generating functions.

Proof. $\sum_{j=0}^\infty e_j (-t)^j = \prod_{i=1}^\infty (1 - x_i t)$.

$\sum_{j=0}^\infty h_j t^j = \prod_{i=0}^\infty (1 + x_i t + x_i^2 t^2 + \dots) = \prod_{i=1}^\infty \frac{1}{1 - x_i t}$.

So

$$\left(\sum_{j=0}^\infty e_j (-t)^j \right) \left(\sum_{j=0}^\infty h_j t^j \right) = 1.$$

Taking the coefficient of t^k we have $\sum_{j=0}^k (-1)^j e_j h_{k-j} = 0$. ■

Corollary 1.2.2. *e_k is a polynomial in $h_1, h_2, \dots, h_k$.*

Proof. Induct on n. Base case: $e_1 = h_1$.

$e_k = \sum_{j=0}^{k-1} (-i)^{j-1} e_j h_{k-j}$. Inductively, for each e_j for $j < k$ us a polynomial $h_1, h_2, \dots, h_j$, so we win. ■

The Hall inner product

This is a positive definite symmetric bilinear pairing

Computation of last class

$$h_\lambda = \sum_\mu A_{\lambda\mu} m_\mu$$

work out the matirx to expand h and m s

Theorem 1.2.1.

$$A_{\lambda m} = \#\{\mathbb{Z}_{\geq 0} matrox B with row sum \lambda and column sum \mu\}$$

Corollary 1.2.3. $A_{\lambda m} = A_{\mu\lambda}$

Corollary 1.2.4. $\langle , \rangle$ *is symmetric.*

THe definition of h_λ is all product of x's

$$h_\lambda(x) = h_{\lambda_1}(x)h_{\lambda_2}(x)\ldots = \prod_{i=1} h_{\lambda_i}(x)$$

h_{λ_i} is the sum of all monomial of degree λ_i

$$= \prod_{i=1}\left(\sum_{B_{i1}+B_{i2}+ldots=\lambda_i}(x_1)^{B_{i1}}(x_2)^{B_{i2}}\ldots\right) = \sum_{B_{ij}\geq 0, B_{i1}+B_{i2}+\ldots=\lambda_i}\prod_i (x_1^{B_{i1}}x_2^{B_{i2}}\ldots)(distributive)$$

$$= \sum_{B_{ij}\geq 0, B_{i1}+B_{i2}+\ldots=\lambda_i}\left(x_1^{\sum_i B_{i1}}\ldots\right) = \sum_{B, rowsum(B)=\lambda} x^{colsum(B)}$$

So coefficient of x^μ is number of B with row sum $= \lambda$ and column sum $= \mu$.

Let's redo this computation using generating function, since we will do a lot of things from gen function perspective.

Start with generating function for h_k

$$\sum_{k=0}^{\infty} t^k h_k(y) = \prod_j \frac{1}{1-ty_j}$$

Multiply copies of it (t renamed to x_i):

$$\prod_i \prod_j \frac{1}{1-x_iy_j} = \prod_i \sum_{k_i}^{\infty} x_i^{k_i} h_{k_i}(y)$$

use distributive law we have

$$= \sum_{k_1,k_2,\ldots=0}^{\infty}\prod_{i=1}^{\infty} x_i^{k_i}h_{k_i}(y_i) = \sum_{k_1,k_2,\ldots=0}^{\infty}(x_1^{k_1}\ldots)h_{k_1}(y_1)h_{k_2}(y_2)\ldots = \sum_{\lambda_1\geq\lambda_2} m_\lambda(x)h_\lambda(y)$$

THe conclusion is that

$$\prod_{i,j=1}^{\infty} \frac{1}{1-x_ix_j} = \sum_{\lambda} m_\lambda(x)h_\lambda(y)$$

plugin the def we have

$$\sum_{\lambda,\mu} A_{\lambda\mu} m_\lambda(x) m_\mu(y)$$

to check $A_{\lambda\mu}$ is symmetric

The product $\prod_{i,j=1}^{\infty} \frac{1}{1-x_ix_j}$ the product of sum of geometric series ... if I expand I just choose the exponents to raise

As computation gets messier and messier gen function methods will be more useful

we will see $\prod_{i,j=1}^{\infty} \frac{1}{1-x_ix_j}$ coming up and up again (we'll call it Cauchy's product since it does not seem to have a name)

We expand a symmetric thing asymmetrically. In general, if we have some identity that can be expanded in this way want can be deduced?

Suppose $p_K(X)$ and $q_L(y)$ are two families of homogeneous symmetric polynomials indexed by some index sets.

Suppose we have an expansion formula

$$\prod_{i,j=1}^{\infty} \frac{1}{1-x_ix_j} = \sum_{K,L} B_{KL} p_K(x) q_L(y), B_{KL} \in \mathbb{Z}$$

Proposition 1.2.1. *From the above condition we can deduce that the $p_K \mathbb{Z}$-span Λ, as do the q_L's.*

If $p_k, q_L \in \mathbb{Q}\Lambda$, and $B_{KL} \in \mathbb{Q}$, then p_k must $\mathbb{Q}$-span Λ, as do the q_L's.

behind the scene (using ... to express Lambda tensor Lambda)

$\langle f, \rangle : \Lambda \to \mathbb{Z}$, it also induces a map $\langle f, \rangle \otimes \mathrm{id} : \Lambda \otimes_{\mathbb{Z}} \lambda \to \Lambda$. (thinking about coefficients)

Lemma 1.2.3.

$$\left\langle f(x), \prod_{i,j} \frac{1}{1-x_ix_j} \right\rangle_{in\ variable\ x} = f(y)$$

Proof. it is enough to check this for f in a $\mathbb{Z}$-basis of Λ.

$$\left\langle h_\lambda(x), \prod \frac{1}{1-x_ix_j} \right\rangle = \left\langle h_\lambda(x), \sum_\lambda m_\lambda(x)h_\lambda(y) \right\rangle = h_\lambda(y)$$

■

Let's go back to the proposition.

Proof. For any $f \in \Lambda$,

$$f(y) = \left\langle f(x), \prod \frac{1}{1 - x_i x_j} \right\rangle$$

$$= \sum_{K,L} B_{KL} p_K(x) q_L(y) = \sum_{K,L} B_{KL} \langle f(x), p_k(\lambda) \rangle q_L(y)$$ ■

Corollary 1.2.5. *Suppose p_λ and q_λ are the families of homogeneous symmetric polynomials indexes by partitins with $\deg p_\lambda = \deg q_\lambda = |\lambda|$ and*

$$\prod \frac{1}{1 - x_i x_j} = \sum B_{\lambda\mu} p_\lambda(x) q_\mu(y), B_{\lambda\mu} \in \mathbb{Z}$$

Then

$$\Lambda = \bigoplus_\lambda \mathbb{Z} p_\lambda = \bigoplus \mathbb{Z} q_\lambda$$

Solve this and get tenure:

$$\prod \frac{1}{1 - x_i y_j z_k} = \sum_{\lambda,\mu,\nu} g_{\lambda\mu\nu} s_\lambda(x) s_\mu(y) s_\nu(z)$$

Suppose now $p_\lambda(X)$ and $q_\mu(y)$ are two families of homogeneous symmetric polynomials in Λ indexed by partition with $\deg p_\lambda = \deg q_\lambda = |\lambda|$ and let

$$\prod_{i,j=1}^{\infty} \frac{1}{1 - x_i x_j} = \sum_{\lambda\mu} B_{\lambda\mu} p_\lambda(x) q_\mu(y), B_{\lambda\mu} \in \mathbb{Z}$$

Let $C_{\lambda\mu} = \langle p_\lambda(x), q_\mu(x) \rangle$. Then B and C are inverses.

Proof. $\left\langle p_\nu(y), \prod \frac{1}{1 - x_i x_j} \right\rangle = p_\nu(x)$ So

$$\sum_{\lambda,\mu} B_{\lambda\mu} p_\lambda(x) \langle p_\nu(y), q_\mu(y) \rangle = \sum_{\lambda,\mu} B_{\lambda\mu} C_{\nu\mu} p_\lambda(x)$$

LHS have linear combination of p RHS we have matching up coefficients of two sides

$$\sum_\mu B_{\lambda\mu} C_{\nu\mu} = \begin{cases} 1 & \lambda = \mu \\ 0 & \lambda \neq \mu \end{cases} \implies BC^T = \text{id}$$

■

In particular, if $\prod \frac{1}{1-x_i x_j} = \sum_\lambda c_\lambda p_\lambda(x) p_\lambda(y)$ the p_λ are orthogonal for $\langle , \rangle$.

If we have no coefficients then this results in an orthornormal basis. $\prod \frac{1}{1-x_i x_j} = \sum_\lambda s_\lambda(x) s_\lambda(y)$ (properties of schur polynomials we will discuss next week.)