Artin's books is a little short on fields.

– Michiael Zieve

Today: $\sqrt[3]{2} \neq \sqrt{a_1} + \sqrt{a_2} + \ldots + \sqrt{a_k}, a_i \in \mathbb{Q}, a_i > 0$.

We show that the left-hand side is in a field K $s.t.$ K has dimension divisible by 3 or ∞ as $\mathbb{Q}$ vector space, while the right-hand side has dimension 2^ℓ.

Definition. Given a field K, a field L containing K is called an *extension* of K, and $L/_K$ (not a quotient) is a "field extension."

Definition. If $L/_K$ is a field extension then its *degree* is $\dim_K L := \dim L$ as K vector space.

Let $L/_K$ be a field extension and let $\alpha \in L$. Then $K(\alpha)$ denotes the smallest field that contains K and $\alpha = \left\{ \frac{a(\alpha)}{b(\alpha)} : a, b \in K[x], b(\alpha) \neq 0 \right\}$.

Let $S = \{f(x) \in K[x] : f(\alpha) = 0\}$. Then S is an ideal of $K[x]$. Since $K[x]$ is PID, $S = (m(x))$ for some $m(x) \in K[x]$. We may assume $m(x)$ is monic or 0. (Clearly m is irreducible.)

Definition. α is *algebraic* over K if $f(\alpha) = 0$ for some nonzero $f(\alpha) \in K[x]$. In this case, the *minimal polynomial* of α over k is $\mathrm{minpol}_K(\alpha) = \mathrm{irr}_K(\alpha) =$ the unique monic irreducible $m(x) \in K[x]$ $s.t.$ $m(\alpha) = 0$.

α is transcendental over K is α is not algebraic over K.

If α is transcendental over K then $K[x] \xhookrightarrow{\text{eval}} K[\alpha] \subseteq K(\alpha)$ this homomorphism extends to an injective homomorphism $K(x) \hookrightarrow K(\alpha)$ which is an isomorphism $K(x) \cong K(\alpha)$.

Now suppose α is algebraic over K with $m(x) = \mathrm{minpol}_K(\alpha)$. Then $K[x] \to K[\alpha]$ is surjective with kernel $(m(x))$. So $K[x]/(m(x)) \cong K[\alpha]$. But $(m(x))$ is maximal $\implies K[\alpha]$ is a field $\implies K(\alpha) = K[\alpha]$.

$K[x]/m(x)K[x]$ as a K-vector space has basis $\{1, x, x^2, \ldots, x^{n-1}\}$ where $n := \deg(m(x))$ (since every coset $K[x]/(m(x))$ contains a unique polynomial of degree $< n$.)

So if α is algebraic over K, and $n := \deg(\mathrm{minpol}_K(\alpha))$, then $K(\alpha)$ has basis $1, \alpha, \ldots, \alpha^{n-1}$ as a K-vector space $\implies \dim_K(K(\alpha)) = n = \deg(\mathrm{minpol}_K(\alpha))$.

Notation: if $L/_K$ is a field extension, then $[L : K] = \dim_K L = \dim L$ as a K vector space.

Example. $[\mathbb{Q}(i) : \mathbb{Q}] = 2$.

$[\mathbb{Q}(\sqrt[n]{2}) : \mathbb{Q}] = n$. (since $x^n - 2$ is irreducible)

$[\mathbb{Q}(e^{2\pi i/p}) : \mathbb{Q}] = p - 1$ if p is prime since $\mathrm{minpol}_{\mathbb{Q}}(e^{2\pi i/p}) = x^{p-1} + x^{p-2} + \ldots + 1$.

Example. Any 3-dimensional $\mathbb{C}$-vector space has dimension 6 as an $\mathbb{R}$-vector space: if

$\alpha_1, \alpha_2, \alpha_3$ is a $\mathbb{C}$-basis, then every element of the vector space can be written in exactly one way as $(a_1 + ib_1)\alpha_1 + (a_2 + ib_2)\alpha_2 + (a_3 + ib_3)\alpha_3$ with $a_j, b_j \in \mathbb{R}$ i.e. as $a_1\alpha_1 + ib_1\alpha_1 + a_2\alpha_2 + ib_2\alpha_2 + a_3\alpha_3 + ib_3\alpha_3$. So $\alpha_j, i\alpha_j$'s is an $\mathbb{R}$-basis of V.

Proposition. *More generally, if $L/_K$ is a field extension and V is an L-vector space of nonzero dimension. Then $\dim_K V = [L : K] \dim_L V$.*

Proof. Let $\alpha_1, \alpha_2, \dots$ be an L-basis for V. Let $\beta_1, \beta_2, \dots$ be a K-basis for L. We'll show that $\{\alpha_i\beta_j\}$ is a K-basis for V.

Every element of V can be written in exactly one way as $\sum_i \ell_i\alpha_i$ with $\ell_i \in L$ and each ℓ_i can be written in exactly one way as $\ell_i = \sum_j k_{ij} beta_j$ with $k_{ij} \in K$.

Every element of V can be written in exactly one way as $\sum_i \sum_j k_{ij}\beta_j\alpha_i \implies \{\alpha_i\beta_j\}$ is a K-basis for V. ■

Corollary. *If K, L, M are fields with $M \supset L \supset K$ then $[M : K] = [M : L] \cdot [L : K]$.*

Example. Eisenstein $\implies x^3 - 2$ irreducible over $\mathbb{Q} \implies [\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3 \implies$ if K is a field containing $\sqrt[3]{2}$ then $[K : Q]$ is divisible by 3 (or ∞.)

If $L/_K$ is an extension and $\alpha \in K$ has a square root in L, then $[K(\sqrt{a}) : K] = 1$ or 2.

So if $a_1, \dots, a_n \in \mathbb{Q}$ then $[\mathbb{Q}(\sqrt{a_1}) : \mathbb{Q}] = 1$ or 2, $[\mathbb{Q}(\sqrt{a_1}, \sqrt{a_2}) : \mathbb{Q}(\sqrt{a_1})] = 1$ or 2, ... Hence if $K_i := \mathbb{Q}(\sqrt{a_1}, \dots, \sqrt{a_n})$ then $[K_{i+1} : K_i] = [K_i(\sqrt{a_{i+1}}) : K_i] = 1$ or 2.

This shows that $[K_i : \mathbb{Q}]$ divides $2^i \implies$ not divisible by 3.

We can also get that $\sqrt[3]{2} \neq$ sum of nested square roots.

You know that a protractor is? WOW! I didn't know what computer was when I was in school. I thought there must be some trade-off!

– Micheal Zieve