# Math 494

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$\mathbb{Z}$ is similar to $\mathbb{Z}/p\mathbb{Z}[x]$ and also to $\mathbb{C}[x]$. So there are analytic statements we can prove algebraically, and then apply it to different coefficient brackets.

**Theorem** (Integral domain is contained in a field)**.** *If $R$ is an integral domain, then $\exists$ injective homomorphism $\varphi : R \to K$ for some field $K$.*

*Proof.* (Analogous to construction of rational numbers) Let $\operatorname{Frac} R = \{(a, b) : a, b \in R.b \neq 0\}$. Write $\frac{a}{b}$ for $(a, b)$. Say $\frac{a}{b} \sim \frac{c}{d}$ if $ad = bc$.

Define $\frac{a}{b} + \frac{c}{d} := \frac{ad+bc}{bd}$ and check that the definition doesn't depend on the choice of representitive for each equivalence class i.e. if $\frac{a}{b} = \frac{A}{B}$ and $\frac{c}{d} = \frac{C}{D}$ then $\frac{ad+bc}{bd} = \frac{AD+BC}{BD}$.

In fact $\operatorname{Frac} R$ is a ring with $0$ element $\frac{0}{1}$ and $1$ element $\frac{1}{1}$, $-\left(\frac{a}{b}\right) = \frac{-a}{b}$.

So there is injection $\mathbb{R} \hookrightarrow \operatorname{Frac} R, r \mapsto \frac{r}{1}$.

Also $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a} \implies \operatorname{Frac} R$ is a field. ■

$\operatorname{Frac} R$ is the "field of fractions" of $R$, or the "fractional field" of $R$. In fact it is the smallest field containing $R$.

"Mapping property": If $R$ is integral domain and $K$ is a field. $\varphi : R \to K$ is injective homomorphism. $R \hookrightarrow \operatorname{Frac} R \dashrightarrow K, \varphi = \iota \circ \theta$.

**Example.**
- $\operatorname{Frac} \mathbb{Z} = \mathbb{Q}$.
- $K =$ field $\implies \operatorname{Frac}(K[x]) = K(X)$.
- $\operatorname{Frac}(\mathbb{Z}[x]) = \mathbb{Q}(x)$.

$\mathbb{C}[x, y]/(xy - 1) \cong \mathbb{C}\left[x, \frac{1}{x}\right]$.

**Definition.** A <u>maximal ideal</u> $M$ of a ring $R$ is an ideal $M \neq R$ *s.t.* $\nexists$ ideal $I$ of $R$ with

$M \subsetneq I \subsetneq R$.

**Example.**

- $R = \mathbb{Z}$, the maximal ideals are $(p)$ where $p$ is prime.
- $R = \mathbb{C}[x]$, the maximal ideals are $(x - \alpha)$, $\alpha \in \mathbb{C}$.

**Lemma.** *If $\varphi : R \to R'$ is a surjective ring homomorphism, then $\ker(\varphi)$ is a maximal ideal if and only if $R'$ is a field.*

*Proof.* Correpondence theorem says that $\ker(\varphi)$ is maximal iff the only ideals of $R$ containing $\ker(\varphi)$ are $(1)$ and $\ker(\varphi)$, and $\ker(\varphi) \neq (1)$ iff the only ideals of $R'$ are $(0)$ and $(1)$, where $(0) \neq (1)$ iff $R'$ is a field. ∎

**Corollary.** *An ideal $I$ of $R$ is maximal $\iff$ $R/I$ is a field.*

**Corollary.** *The ideal $(0)$ of $R$ is maximal $\iff$ $R$ is a field.*

**Lemma.** *Let $K$ be a field.*

1. *The maximal ideals of $K[x]$ are $(f(x))$ with $f(x)$ irreducible.*
2. *If $\varphi : K[x] \to R'$ is a homomorphism to an integral domain $R'$, then $\ker(\varphi)$ is either $(0)$ (which implies the map is injective) or a maximal ideal.*

$fg \in \ker(\varphi) \implies \varphi(fg) = \varphi(f)\varphi(g) = 0 \implies \varphi(f) = 0$ or $\varphi(g) = 0 \implies f \in \ker(\varphi)$ or $g \in \ker(\varphi)$.

But $\ker(\varphi) =$ ideals of $K[x] = (h)$ for some $h \implies \ker(\varphi) = (0)$ or $(h)$, $h$ irreducible, or $(1)$ (impossible sime $R'$ is integral domain).

<u>HILBERT'S NULLSTELLENSTAZ</u> The maximal ideals of

$$R := \mathbb{C}[x_1, \ldots, x_n]$$

are

$$(x_1 - \alpha_1, x_2 - \alpha_2, \ldots, x_n - \alpha_n)$$

with $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. (So they are in bijection of $\mathbb{C}^n$.)

Note: this ideal is the kernel of the evaluation homomorphism

$$\mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}, \quad f(x_1, \ldots, x_n) \mapsto f(\alpha_1, \ldots, \alpha_n).$$

This is maximal since image of a homomorphism is a field.

*Proof.* Let $M$ be a maximal ideal of $R$. Consider quotient map $\pi : R \to R/M$. We have $M = \ker \pi$. It suffice to show that $M$ contains $x_1 - \alpha_1, \ldots, x_n - \alpha_n$ for some $\alpha_1, \ldots, \alpha_n$.

Restrict this to the subring $\mathbb{C}[x_1] \subseteq R$ to get $\mathbb{C}[x_1] \to R/M$. Since $R/M$ is a field, then the kernel of $\mathbb{C}[x_1] \to R/M$ is either $(0)$ or a maximal ideal of $\mathbb{C}[x_1]$ $i.e.$ $(x_1 - \alpha_1)$.

It cannot be $(0)$ since if it were $(0)$ then it was a injection and we get $\mathrm{Frac}(\mathbb{C}[x_1]) \hookrightarrow R/M$. But these maps are the identity on $\mathbb{C}$, so get injective $\mathbb{C}$-linear map.

$R/M$ is a countable dimensional $\mathbb{C}$-vector space since it is spanned by $x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n}$. $\mathbb{C}(x)$ has uncountable dimension as $\mathbb{C}$-vector space since $\frac{1}{x-\alpha}, \alpha \in \mathbb{C}$ are linearly independent.

So we obtain a injective map from uncountable dimensinoal $\mathbb{C}$ -vector space to a countable dimensional $\mathbb{C}$-vector space, a contradiction.

So $M$ has to be $(x_1 - \alpha_1)$. This applies to restriction to any $\mathbb{C}[x_i]$'s. ■