Math 494

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Introduction and Administration

Two highlights of the course:

1. For a quadratic equation $ax^2 + bx + c = 0 (a \neq 0)$ there are roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

   There is similar formulas for polynomials of degree 3 and 4, but not degree 5 or higher.

2. $x^2 + 3 = y^3$ has only integer solutions $x = \pm 5, y = 3$. Important and powerful methods behind simple problems.

- HW due Monday midnight
- Office hours: Sunday 2 - 3:30 p.m., Friday 7 - 8:30 p.m.

Rings

Review

Definition. A group $(G, *)$ is a set G equipped with function $* : G \times G \to G$ and element $1_G \in G$ $s.t.$

- $*$ is associative
- 1_G is the identity element (hence unique)
- $\forall g \in G, \exists h \in G$ $s.t.$ $gh = hg = 1_G$.

Note that the identity and inverse elements are unique.

We say G is abelian if $ab = ba, \forall a, b \in G$.

Definition. Group homomorphism

Lemma. *If G, H are groups and H is abelian, then $\mathrm{Hom}(G, H)$ is an abelian group under the operation*

$$\varphi + \psi : G \to H, g \mapsto \varphi(g) + \psi(g).$$

Definition. A ring is a set R with two functions $+, \cdot : R \times R \to R$ *s.t.* :

- $(R, +)$ is an abelian group (identity $0 \in R$, inverse element $-r$ for every $r \in R$).
- $(R, \cdot)$ is associative with an identity 1.
- Distribution laws hold

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (b + c) \cdot a = b \cdot a + c \cdot a.$$

Hence if G is an abelian group, then $\mathrm{End}(G)$ is a ring.

Definition. Let R, S be rings. A ring homomorphism is a function $f : R \to S$ which preserves $+, \cdot, 0, 1$, additive inverses. (In fact it suffices to preserve $+, \cdot, 1$ since f preserves $+$ implies it preserves 0 and inverses.)

$$f(r +_R r') = f(r) +_S f(r')$$
$$f(r \cdot_R r') = f(r) \cdot_S f(r')$$
$$f(1_R) = 1_S.$$