Math 493

Yiwei Fu

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1 Representation Theory, cont'd

Lemma 1.1. *If $\rho : G \to \mathrm{GL}(V)$ is a finite dimensional G-representation of a finite group G and χ is the character of ρ which is non trivial and irreducible, then*

$$\sum_{g \in G} \chi(g) = 0,$$

and the multiplicity of the trivial representation in any decomposition of ρ as the sum of irredicuble representations is

$$\langle \chi_{triv}, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g).$$

Theorem 1.1. *If $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(V')$ are irreducible representations of G with characters χ and χ', then*

$$\langle \chi, \chi' \rangle = \begin{cases} 1 & \rho \cong \rho', \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. This says

$$\frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\chi'(g)} = \begin{cases} 1 & \rho \cong \rho', \\ 0 & \text{otherwise}. \end{cases}$$

From homework we have character of $\mathrm{Hom}_{\mathbb{C}}(V, W)$ is $\chi \overline{\chi'}$. So this is the multiplicity of ρ_{triv} in $\mathrm{Hom}_{\mathbb{C}}(V, W)$.

$\mathrm{Hom}_{\mathbb{C}}(V, W)$ has structure of G-representation via $g.\phi$ mapping $v \mapsto g.\phi(g^{-1}.v)$. ϕ is fixed by G in the sense that $\forall g, g.\phi(g^{-1}.v) = \phi(v)$.

So $\phi : V \to W$ is a homomorphism of G-representations. ∎

Lemma 1.2. *Let $f : G \to \mathbb{C}$ be a class function (i.e. a function which is constant on each conjugacy class of G.)*

Let $\rho : G \to \mathrm{GL}(V)$ be a representation. Let $\phi : V \to V$ be $\phi = \sum_{g \in G} f(g)\rho(g)$.

If ρ is irreducible of degree n with character χ, then ϕ is scaling by $\frac{1}{n}\sum_{g\in G} f(g)\chi(g) = \frac{|G|}{n}\langle f, \overline{\chi}\rangle$.

Proof. ϕ is a $\mathbb{C}$-linear map $V \to V$. We want to show

$$\forall g \in G, \phi\rho(g) = \rho(g)\phi. \tag{1}$$

This is true since

$$\begin{aligned}
\phi\rho(g) &= \sum_{g' \in G} f(g')\rho(g')\rho(g) \\
&= \sum_{g' \in G} f(g')\rho(g'g) \\
&= \sum_{h \in G} f(hg^{-1})\rho(h) \\
\rho(g)\phi &= \sum_{g' \in G} f(g')\rho(g)\rho(g') \\
&= \sum_{h \in G} f(g^{-1}h)\rho(h).
\end{aligned}$$

But $g^{-1}h = h^{-1}(hg^{-1})h$ so $f(g^{-1}h) = f(hg^{-1})$.

Hence ϕ is a homomorphism of G-representations from $V \to V$, so it is scaling, say by c. Then we have

$$c \dim V = \mathrm{tr}(\phi) = \sum_{g\in G} f(g)\,\mathrm{tr}(\rho(g)) = \sum_{g \in G} f(g)\chi(g).$$

■

Theorem 1.2. *The characters $\chi_1, \ldots, \chi_n$ of the irreducible representations of G form an orthonormal basis of the space of class functions on G.*

Proof. Just need to show that χ_i's span the space of class functions.

Pick any class function f. Replace f by $f - \sum_{i=1}^{n} \langle f, \chi_i\rangle \chi_i$ to assume f is orthogonal to very χ_i. We want to show that $f = 0$.

From Lemma 1.2 we have that $\forall i$, the ϕ coming from χ_i is 0.

Hence $\forall$ representation ρ, the ϕ coming from ρ is 0. Apply this to the regular representation ρ_{reg}.

$\rho =$ regular representation $\iff V$ has basis $e_g, g \in G$.

$$\begin{aligned}\phi(e_1) &= \sum_{g\in G} f(g)\rho(g)(e_1) \\ &= \sum_{g \in G} f(g) e_g.\end{aligned}$$

But ϕ is the 0 map, so

$$0 = \sum_{g\in G} f(g) e_g.$$

Hence $f(g) = 0, \forall g$. ■

Proposition 1.1. *For $g \in G$, let $c(g)$ be the size of the conjugacy class of G. Then if $\chi_1, \dots, \chi_n$ are the irreducible characters of G then*

$$\sum_{i=1}^{n} \overline{\chi_i(g)} \chi_i(g) \frac{|G|}{c(g)} = |C_G(g)|$$

and if $g' \in G$ is not conjugate to g then

$$\sum_{i=1}^{n} \overline{\chi_i(g)} \chi_i(g) = 0.$$

Proof. Let $f : G \to \mathbb{C}$ be 1 on the conjugacy class of g, and 0 elsewhere. This is a class function. Then

$$f = \sum_{i=1}^{n} \langle f, \chi_i \rangle \chi_i$$

where

$$\langle f, \chi_i \rangle = \frac{1}{|G|} \sum_{g'' \sim g} \chi_i(g'') = \frac{c(g)}{|G|} \overline{\chi_i(g)}.$$

Hence

$$f = \frac{c(g)}{|G|} \sum_{i=1}^{n} \overline{\chi_i(g)} \chi_i$$

So for $g' \in G$ we have

$$f(g') = \frac{c(g)}{|G|} \sum_{i=1}^{n} \overline{\chi_i(g)} \chi_i(g')$$

If g is conjugate to g' then $\chi_i(g') = \chi_i(g) \implies \sum \overline{\chi_i(g)} \chi_i(g) = \frac{|G|}{c(g)}$.

If g is not conjugate to g' then $\sum \overline{\chi_i(g)}\chi_i(g) = 0$. ■