Math 493

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1 Representation Theory, cont'd

RECALL: If $A \in M_{n\times n}(F)$ a n-by-n matrix then the trace of A (denoted as $\operatorname{tr}(A)$) is the sum of the diagonal entries of A.

Key properties:

$$\operatorname{tr}(AB) = \operatorname{tr}(BA) \implies \operatorname{tr}(C^{-1}AC) = \operatorname{tr}(A).$$

Frobenius did representation theory without matrices, how?

AMAZING FACT: we lose no information by replacing a finite-dimensional complex representation $\rho : G \to \mathrm{GL}(V)$ by its "character" $\chi : G \to \mathbb{C}, g \mapsto \operatorname{tr}(\rho(g))$.

Precisely: Two representations ρ_1, ρ_2 are isomorphic if and only if their characters are equal (as functions $G \to \mathbb{C}$).

FACT:

$$\chi(hgh^{-1}) = \operatorname{tr}(\rho(hgh^{-1})) = \operatorname{tr}(\rho(h)\rho(g)\rho(h)^{-1}) = \operatorname{tr}(\rho(g)) = \chi(g).$$

So χ is a "class function", meaning a function $G \to \mathbb{C}$ which is constant on each conjugacy class of G.

NOTE: if V_1, V_2 are finite-dimensional G-representations, then the character of $V_1 \oplus V_2$ is (char. of V_1) + (char. of V_2).

So the character of any finite-dimensional representation is the sum of the characters of finitely many irreducible representations.

GREAT FACT(for finite-dimensional complex representations of a finite group G): the characters of the irreducible representations of a finite group G form a basis for the space of class functions on G. This implies that the number of irreducible representations of G (up to $\cong$) equals the number of conjugacy class of G.

Definition 1.1. Given

$$\phi : G \to \mathbb{C}, \psi : G \to \mathbb{C},$$

the inner product on functions $G \to \mathbb{C}$ is defined as

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\psi(g)}.$$

So

$$\begin{aligned}
\langle \phi, \psi \rangle &\in \mathbb{C}, \\
\langle \phi + c\phi', \psi \rangle &= \langle \phi, \psi \rangle + c\langle \phi', \psi \rangle, \\
\langle \phi, \psi + c\psi' \rangle &= \langle \phi, \psi \rangle + \bar{c}\langle \phi, \psi' \rangle, \\
\langle \phi, \phi \rangle &\in \mathbb{R}_{\geq 0}, \\
\langle \phi, \phi \rangle = 0 &\iff \phi = 0.
\end{aligned}$$

<u>GREATER FACT</u>: Under the inner product defined above, irreducible characters (= characters of irreducible representations) forms an <u>orthonormal basis</u> for the space of class functions on G.

If G acts on a finite set S, then the corresponding linear representation $\rho : G \to \mathrm{GL}(\mathbb{C}^{|}S|)$ has character χ where $\chi(g) = \#$ of fixed points of g on S.

The character of the regular representation:

$$\chi(g) = \begin{cases} 0 & \text{if } g \neq 1, \\ |G| & \text{if } g = 1. \end{cases}$$

Irreducible character of S_3:

- Trivial representation:

  $$\rho : G \to \mathbb{C}^*, g \mapsto 1.$$

  We have $\chi_0 = 1$.

- Sign representation:

  $$\rho : G \to \mathbb{C}^*, g \mapsto \operatorname{sign}(g).$$

  χ_s maps $(\mathbf{1\ 2\ 3}) \mapsto 1, (\mathbf{1\ 2}) \mapsto -1, (\mathbf{1}) \mapsto 1.$

- 2-dimensional representation:

$$\rho : G \to \mathrm{GL}(V), V = \{(a,b,c) \in \mathbb{C}^3, a+b+c=0\}, g \cdot (a,b,c) = (c,a,b).$$

The character χ satisfies:

$$\chi + 1 = \chi_{reg}.$$

χ maps $(\mathbf{1\ 2\ 3}) \mapsto -1, (\mathbf{1\ 2}) \mapsto 0, (\mathbf{1}) \mapsto 2.$

$$\begin{aligned}
(\chi_0, \chi_0) &= \frac{1}{|G|} \sum_{g \in G} \chi_0(g) \overline{\chi_0(g)} \\
&= \frac{1}{|G|} \sum_{g \in G} 1 \\
&= 1. \\
(\chi, \chi) &= \frac{1}{6}(1+1+0+0+0+4) = 1. \\
(\chi_0, \chi) &= \frac{1}{|G|} \sum_{g \in G} \chi_0(g) \overline{\chi(g)} \\
&= \frac{1}{6}(2(-1) + 3(0) + 1(2)) \\
&= 0. \\
(\chi_s, \chi_s) &= \frac{1}{6}(2(1) + 3(-1 \cdot (-1)) + 1(1 \cdot 1)) = 1. \\
(\chi_s, \chi) &= \frac{1}{6}(2(-1) + 1(2)) = 0.
\end{aligned}$$