Math 493

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Nov 2, 2021

1 Representation Theory, cont'd

<u>RECALL</u>: a linear representation of a group G is a homomorphism $\rho : G \to \mathrm{GL}(W)$ for some vector space V.

Say ρ is <u>irreducible</u> if V has not subrepresentations except $\{0\}$ and V, where a subrepresentation is a subspace W of V $s.t.$ $\rho(g)(W) \subseteq W, \forall g \in G$. (so that ρ induces a homomorphism: $G \to \mathrm{GL}\,(W)$.)

1-dimensional representations:

$$\rho : G \to \mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^*.$$

If G is finite then $\rho(G)$ is a finite subgroup of $\mathbb{C}^*$, hence it is cyclic. So ρ is a homomorphism from G ot a cyclic group.

Theorem 1.1. *(Maschke's theorem) Every finite dimensional complex representation of a finite group G can be written as a direct sum of irreducible subrepresentations.*

Remark. It's like saying there is a prime factorization. Given homomorphism: $\rho : G \to \mathrm{GL}(V)$, we can write $V = W_1 \oplus \ldots \oplus W_k$ with W_i subspaces of V such that (ρ, W_i) is an irreducible subrepresentation of V.

This result follows from:

Theorem 1.2. *If $\rho : G \to \mathrm{GL}(V)$ is a finite dimensional complex representation of a finite group G, and W is a subrepresentation, then there exists subrepresentation W' of V such that $V = W \oplus W'$.*

Remark. Same proof works for any field K such that $|G|$ is invertible in K.

Proof for Theorem 1.2. Pick any "projection map" $\pi : V \to W$, meaning a linear map $V \to W$ which restricts to the identity map on W. (e.g. extend a basis of W to a basis of V

and define π to be identity on the basis of W, and map to any chosen vectors in W on the basis vectors outside of W.)

Define:

$$\phi : V \to W,\ v \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot v)$$

<u>Claim</u>: ϕ is G-invariant projection map from $V \to W$.

Check:

(1) $\phi|_W = \mathrm{id}_W$: If $w \in W$ then $g^{-1} \cdot w \in W$ since W is a subrepresentation. Then

$$\pi|_W = \mathrm{id}_W \Rightarrow \pi(g^{-1} \cdot w) = g^{-1} \cdot w \Rightarrow g \cdot \pi(g^{-1} \cdot w) = g(g^{-1} \cdot w) = w.$$

So $\phi(w) = \frac{1}{|G|} \sum_{g \in G} w = w$.

(2) Clearly it $\phi(v) \in W$ since $\pi(g^{-1} \cdot v) \in W$, and then $g \cdot \pi(g^{-1} \cdot v) \in W$ because W is a subrepresentation.

(3) ϕ is a linear map.

$$\phi(v + v') = \phi(v) + \phi(v')$$

since ϕ is a linear combination of linear maps.

(4) Finally show that ϕ is G-invariant: for $h \in G$,

$$\begin{aligned} h \cdot \phi(v) &= h \cdot \left(\frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot v) \right) \\ &= \frac{1}{|G|} \sum_{g \in G} h \cdot (g \cdot \pi(g^{-1} \cdot v)) \\ &= \frac{1}{|G|} \sum_{g \in G} g' \cdot \pi((g')^{-1} h \cdot v) \quad (g' = hg, (g')^{-1} = g^{-1}h^{-1}.) \\ \iff \phi(h \cdot v) &= \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} h \cdot v). \end{aligned}$$

This proves the claim.

Let $W' := \ker(\phi)$. Then W' is a subrepresentation of V. Also since $\phi|_W = \mathrm{id}_W$ and $\phi(V) = W$, we have $V = W \oplus W'$. ■

Theorem 1.3. *If V is a finite dimensional complex representation of a finite group G, then V can be written in exactly one way as an (internal) direct sum $V = V_1 \oplus \ldots \oplus V_k$. where each V_i is a direct sum of (one or more) copies of an irreducible subrepresentation W_i and $W_i \ncong W_j, \forall i \neq j$.*

Said another way:

If $V = U_1 \oplus \ldots \oplus U_\ell = R_1 \oplus R_2 \oplus \ldots \oplus R_m$ with U_i, R_j irreducible subrepresentations of V. then $\forall i$ the number of U_i's isomorphic to U_i equals the number of R_j's isomorphic to U_i, and the direct sum of those U_j's equals the direct sum of those R_j's.

Proof. We have a lemma first.

Lemma 1.1. *A homomorphism $\phi : V \to W$ between irreducible G-representations is either 0 or an isomorphism.*

Proof of lemma. $\ker \phi$ is a subrepresentation of V. It is either 0 of V.

$\mathrm{Im}(\phi)$ is a subrepresentation of W. It is either 0 or W.

Now say

$$V = U_1 \oplus \ldots \oplus U_\ell = R_1 \oplus \ldots \oplus R_m$$

with U_i, R_j's irreducible subrepresentations of V.

Consider

$$U_1 \xhookrightarrow{\text{inclusion}} V \xrightarrow{\text{coord. proj}} R_i.$$

This is a homomorphism of irreducible G-representations.

By lemma, it is 0 or an isomorphism. It cannot be 0 $\forall i$ since the image of $U_1 \hookrightarrow V$ is not $\{0\}$.

So there exists i $s.t.$ $U_1 \hookrightarrow V \twoheadrightarrow R_i$ is an isomorphism of G-representations.

Relabel to assume $i = 1$. Continue to get that the the set of U_i's up to isomorphism, equals the set of R_j's, up to isomorphism.

Rewrite: $V = U_1^{a_1} \oplus U_2^{a_2} \oplus \ldots \oplus U_\ell^{a_\ell}$ where $a_i > 0$, U_i is irreducible and $U_i \ncong U_j$ for any $i \neq j$.

Then the R_i decomposition becomes $V = R_1^{b_1} \oplus \ldots \oplus R_m^{b_m}$ where $b_i > 0, R_i \cong U_i$ as G-representations.

Consider

$$U_1^{a_1} \hookrightarrow V \twoheadrightarrow R_2^{b_2} \oplus \ldots \oplus R_m^{b_m}.$$

Checking lemma gives that this is 0, since

$$U_1 \hookrightarrow U_1^{a_1} \hookrightarrow V \twoheadrightarrow R_2^{b_2} \oplus \ldots \oplus R_m^{b_m} \to W_j$$

is a homomorphism between different irreducible representations (so it is 0).

Hence

$$U_1^{a_1} \subseteq \ker(\text{Projection of } R_1^{b_2} \oplus \ldots \oplus R_m^{b_m} \text{ onto } R_2^{b_2} \oplus \ldots \oplus R_m^{b_m}) = R_1^{b_1}.$$

Similarly $U_1^{a_1} \supseteq R_1^{b_1} \implies U_1^{a_1} = R_1^{b_1} \implies a_1 = b_1$. ■