Math 493

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1 Representation Theory

Definition 1.1. (G-representation) A linear representation of a group G on a vector space V is a (group) homomorphism $\rho : G \to \mathrm{GL}(V)$ where $\mathrm{GL}(V)$ is the group of linear transformations $V \to V$.

We have

$$\deg(p) \overset{\text{def}}{=} \dim(\rho) \overset{\text{def}}{=} \dim V.$$

Example 1.1.

- Trivial representation:

$$\rho(g) = \mathrm{id}_V,\ \forall g \in G.$$

- Representation of C_3 on $V = \mathbb{C}^3$ maps

$$(\mathbf{1\ 2\ 3}) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

- For any action of G on a finite set S, let V be a vector space with basis in bijection with S, say basis $e_s, (s \in S)$, where $\rho(g)$ maps $e_s \mapsto e_{g.s}$.

$$S_3 \to \mathrm{GL}_3(\mathbb{C}),\ (\mathbf{1\ 2\ 3}) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, (\mathbf{1\ 2}) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$D_4 \to \mathrm{GL}_4(\mathbb{C}), (\mathbf{1\ 2\ 3\ 4}) \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, (\mathbf{1\ 2})(\mathbf{3\ 4}) \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Definition 1.2. The regular representation is the representation associated to the action of G on itself by left-multiplication. ($\implies$ dimension $= |G|$.)

Definition 1.3. The sign representation of S_n is

$$\rho : S_n \to \mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^*, \rho(g) = \mathrm{sgn}(g).$$

Definition 1.4. If V is a G-representation (i.e. $\rho : G \to \mathrm{GL}(V)$ is a homomorphism,) then a sub-representation of V is a subspace W of V which is G-invariant, in the sense that

$$g.w \in W,\ \forall g \in G, w \in W\ i.e.\ \rho(g)(W) \subseteq W.$$

A subspace W of V is a sub-representation if and only if the following exists.

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{GL}(V) \\ & \searrow & \uparrow \\ & & \{\psi \in GL(V) : \psi(W) \subseteq W\} \end{array}$$

Example 1.2.
- The trivial representation of G on V acts as the trivial representation on every subspace of V.
- D_4 acts on $\mathbb{R}^2$ via isometries. But no 1-dimensional of $\mathbb{R}^2$ is G-invariant.

  The action of D_4 on C^4 has 1-dimensional invariant subspace $\mathbb{C}(e_1+e_2+e_3+e_4) = V_1$. It induces the trivial representation on this subspace.

  There is also a 3-dimensional invariant subspace $\{c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 : c_1 + c_2 + c_3 + c_4 = 0\} = V_3$.

  Notice that $\mathbb{C}^4 = V_1 \oplus V_3$. Can we decompose further?

  $$v = e_1 - e_2 + e_3 - e_4, (\mathbf{1\ 2\ 3\ 4}) : v \mapsto -v, (\mathbf{1\ 2})(\mathbf{3\ 4})v \mapsto -v$$

  So V_3 has G-invariant subspace $W = \mathrm{span}\{e_1 - e_2 + e_3 - e_4\}$

Orthogonal complement of W in V_3 is

$$\{c_1e_1 + \ldots + c_4e_4 = 0, c_1 + c_2 + c_3 + c_4 = 0, c_1 - c_2 + c_3 - c_4 = 0.\}$$

which is G-invariant. This has no 1-dimensional G-invariant subspace.

If V, W are G-representations then so is $V \oplus W$. via $g.(v, w) = (g.v, g.w)$. In terms of matrices: if $\rho_1 : G \to \mathrm{GL}(V), \rho_2 : G \to \mathrm{GL}(W)$,

$$g \mapsto \begin{bmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{bmatrix}$$

e.g.

$$\begin{aligned} \rho_1 &: \mathbb{C}_2 \to \mathbb{C}^*, g \mapsto 1 \\ \rho_2 &: \mathbb{C}_2 \to \mathbb{C}^*, g \mapsto \mathrm{sign}(g) \\ \rho_1 \oplus \rho_2 &: C_2 \to \mathrm{GL}_2(\mathbb{C}). \end{aligned}$$

For any n-th root of unity ζ,

$$\rho : (\mathrm{Z} / n \, \mathrm{Z}) \to \mathbb{C}^* = \mathrm{GL}_1(\mathbb{C}), i \mapsto \zeta^i$$

is a 1-dimensional representation.

Definition 1.5. A representation is irreducible if it has no proper positive-dimensional sub-representations.

Definition 1.6. Two G-representations V and W are isomorphic if there is a vector space isomorphism $\phi : V \to W$ which is compatible with the G-action, in the sense that $g.\phi(v) = \phi(g.v), \forall g \in G, v \in V$.

In terms of commutative diagrams:

$$\begin{array}{ccc} V & \xrightarrow{v \mapsto g.v} & V \\ \downarrow{\phi} & & \downarrow{\phi} \\ W & \xrightarrow{w \mapsto g.w} & W \end{array}$$